J.D. Jackson Problem 11.6

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The rocket's reference frame here is not inertial, which means that many of the transformations are not as straight-forward as usual. In particular, γ and β are functions of time (either t or t').

1 Velocity in the Earth's frame

Although it is not asked for explicitly, I'll first find an expression for the rocket's velocity as measured in the Earth frame as a function of the rocket's proper time, $v(t')$. It may seem like a strange mixing of prime and unprime quantities, but this result will be useful a few times.

We'll use the results of problem 11.5 to see that

$$a = \frac{(1 - \frac{v^2}{c^2})^{3/2}}{(1 + \frac{v \cdot u}{c^2})^3} a' \tag{1}$$

$u' = 0$ because the rocket does not move in its own frame, so the entire denominator of the previous equation is one. Additionally, we know that $a' = g$ from the problem.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a = \left(1 - \frac{v^2}{c^2}\right)^{3/2} g = \frac{g}{\gamma^3} \tag{2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\gamma \mathrm{d}t'} \tag{3}$$

We now have two expressions for $\frac{\mathrm{d}v}{\mathrm{d}t}$, so let's set them equal to each other.

$$\frac{\mathrm{d}v}{\gamma \mathrm{d}t'} = \frac{g}{\gamma^3} \tag{4}$$

Cross multiplying, we have $\gamma^2 \mathrm{d}v = g\,\mathrm{d}t'$, and we can integrate both sides

$$\int \frac{\mathrm{d}v}{\left(1 - \frac{v(t')^2}{c^2}\right)} = \int gt' \tag{5}$$

$$c \operatorname{arctanh}\left(\frac{v(t')}{c}\right) = gt' \tag{6}$$

$$\frac{v(t')}{c} = \tanh \frac{gt'}{c} \tag{7}$$

$$v(t') = c \tanh \frac{gt'}{c} \tag{8}$$

2 Travel time in Earth frame

Having that preliminary result out of the way, we're now ready to solve for the required quantities. We'll start by integrating the time dilation formula

$$\mathrm{d}t = \gamma \, \mathrm{d}t' \tag{9}$$

$$\int \mathrm{d}t = \int \gamma \, \mathrm{d}t' \tag{10}$$

$$t = \int_0^{t'} \frac{\mathrm{d}t'}{\sqrt{1 - \frac{v^2(t')}{c^2}}} \tag{11}$$

Now we can refer back to our $v(t')$ expression (Eq: 7).

$$t = \int_0^{t'} = \frac{\mathrm{d}t'}{\sqrt{1 - \tanh^2 \frac{gt'}{c}}} \tag{12}$$

$$\boxed{t = \frac{c}{g} \sinh \frac{gt'}{c}} \tag{13}$$

It is, perhaps, more insightful to write the previous equation as

$$\frac{gt}{c} = \sinh \frac{gt'}{c} \tag{14}$$

Or, if we express both times as unitless quantities measured in multiples of c/g[1], then we can write $t = \sinh t'$. We can see now that at launch, $t = t' = 0$ the Earth's clock and the rocket's clock tick at the same speed. But as the rocket continues to accelerate, it's clock apparently slows when observed from Earth.

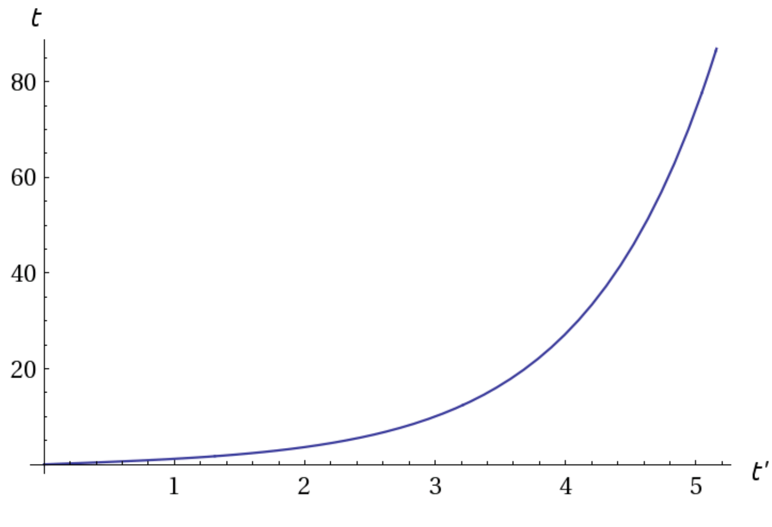

Figure 1: t is related to t' by a sinh function. Plot is in units pf c/g.

[1]Clocking in at 353.8 days, c/g is remarkably close to one year. When $t' = 5$ years, $\frac{gt'}{c} \approx 5$.

3 Distance travelled in the Earth frame

To find the distance travelled we will use the simplest kinematic equation.

$$d = vt \tag{15}$$

We'll write the previous equation in differential form and use the chain rule

$$\mathrm{d}d = v(t')\,\mathrm{d}t' \frac{\mathrm{d}t}{\mathrm{d}t'} \tag{16}$$

$$d = \int v(t')\mathrm{d}t' \frac{\mathrm{d}t}{\mathrm{d}t'} \tag{17}$$

We found t as a function of t', in Eq: 13, so now we can differentiate it to write,

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \cosh\left(\frac{gt'}{c}\right) \tag{18}$$

We will again use our result for $v(t')$ (this time in the form of Eq: 8). So we can now find d.

$$d = \int_0^{t'} c\tanh\left(\frac{gt'}{c}\right)\mathrm{d}t' \cosh\left(\frac{gt'}{c}\right) \tag{19}$$

$$d = c\int_0^{t'} \sinh\left(\frac{gt'}{c}\right)\mathrm{d}t' \tag{20}$$

$$d = \frac{c^2}{g}\left[\cosh\left(\frac{gt'}{c}\right)\right]_0^{t'} \tag{21}$$

$$\boxed{d = \frac{c^2}{g}\left[\cosh\left(\frac{gt'}{c}\right) - 1\right]} \tag{22}$$

4 Some actual numbers

The time for a quarter journey (a single acceleration) is given by

$$t_{1/4} = t(5\,years) = 84.24\,years \tag{23}$$

The total journey is four of these segments

$$T_{total} = 4t_{1/4} = 337\,years \tag{24}$$

Likewise, we can find the quarter journey distance

$$_{1/4} = d(5\,years) = 7.873 \times 10^{17} m \tag{25}$$

$$d_{1/2} = 2d_{1/4} = 1.575 \times 10^{18} m \tag{26}$$