J.D. Jackson Problem 9.11

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We'll start by writing the charge density. We could write it directly in spherical coordinates, but for clarity, we'll begin in the more intuitive Cartesian coordinates.

$$\rho = q\delta(x)\delta(y)[2\delta(z) - \delta(z - a\cos\omega t) - \delta(z + a\cos\omega t)] \tag{1}$$

Now we can convert to spherical coordinates using the various delta function identities. Many useful delta identities can be found at wolfram math world.

$$\rho = \frac{q\delta(\theta)\delta(\phi)}{r^2\sin\theta}[2\delta(r) - \delta(r - a\cos\omega t) - \delta(r + a\cos\omega t)] \tag{2}$$

The problem specifies that $ka \ll 1$ so we can use equation 9.170 from the text to calculate the various Q_{lm} terms.

$$Q_{lm} = \int r^l Y *_{lm} \rho \, \mathrm{d}V \tag{3}$$

Directly substituting the forms of $Y*_{lm}$ and ρ we get the following (where empty brackets indicate the bracked r-based delta functions from ρ).

$$Q_{lm} = \iiint r^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta)e^{-im\phi}\frac{q\delta(\theta)\delta(\phi)}{r^2\sin\theta}[\cdots]r^2\sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{4}$$

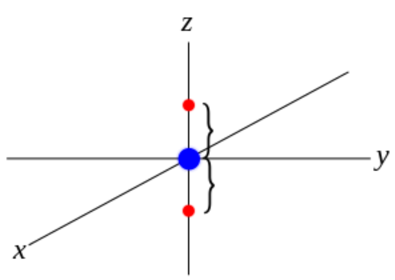
Our charge distribution is azimuthally symetric, so all of the $m \neq 0$ terms will be zero. Looking only at the $m = 0$ terms, the form simplifies nicely.

$$Q_{l0} = \iiint r^l \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) q\delta(\theta)\delta(\phi)[\cdots] \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{5}$$

The θ and ϕ integrals are fairly easy to evaluate because of the delta functions.

$$Q_{l0} = q\sqrt{\frac{2l+1}{4\pi}}\left[2\int_0^\infty r^l\delta(r)\,\mathrm{d}r - \int_0^\infty r^l\delta(r - a\cos\omega t)\,\mathrm{d}r - \int_0^\infty r^l\delta(r + a\cos\omega t)\,\mathrm{d}r\right] \tag{6}$$

Evaluating the integrals in the previous expression must be done carefully as the results will vary depending on the value of l. For the $l = 0$ case the first integral will cancel the second and third. For all



other l the first integral is zero. For odd l the second and third integrals cancel each other, and for even l the second and third terms add together.

So, for the non trivial values of l we see that.

$$Q_{l0} = -2q\sqrt{\frac{2l+1}{4\pi}}a^l \cos^l \omega t \tag{7}$$

The lowest non-trivial Q is

$$Q_{20} = -\sqrt{\frac{5}{\pi}}qa^2 \cos^2 \omega t \tag{8}$$

Using a trig identity, we can rewrite this in another form which emphasizes the fact that the oscilation frequency is actually 2ω.

$$Q_{20} = -\sqrt{\frac{5}{\pi}}qa^2 \frac{1+\cos(2\omega t)}{2} \tag{9}$$

To find the power per solid angle we can use equation 9.151.

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z_0}{2k^2} |a(l,m)|^2 |\mathbf{X}_{lm}|^2 \tag{10}$$

The table on page 437 give us

$$|\mathbf{X}_{20}|^2 = \frac{15}{8\pi} \sin^2\theta \cos^2\theta \tag{11}$$

Finding the $a_E(l,m)$ is not at all obvious. I don't even know if it is supposed to depend on time or not. But I do know the following from equation 9.169.

$$a(l,m) \approx \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}Q_{lm} \tag{12}$$

Alright, I'm not going to waste either of our time pretending I know what to do here.